

# ON CONVEX-CYCLIC OPERATORS

TERESA BERMÚDEZ, ANTONIO BONILLA, AND N. S. FELDMAN

**ABSTRACT.** We give a Hahn-Banach Characterization for convex-cyclicity. We also obtain an example of a bounded linear operator  $S$  on a Banach space with  $\sigma_p(S^*) = \emptyset$  such that  $S$  is convex-cyclic, but  $S$  is not weakly hypercyclic and  $S^2$  is not convex-cyclic. This solved two questions of Rezaei in [23] when  $\sigma_p(S^*) = \emptyset$ . We also characterize the diagonalizable normal operators that are convex-cyclic and give a condition on the eigenvalues of an arbitrary operator for it to be convex-cyclic. We show that certain adjoint multiplication operators are convex-cyclic and show that some are convex-cyclic but no convex polynomial of the operator is hypercyclic. Also some adjoint multiplication operators are convex-cyclic but not 1-weakly hypercyclic.

## 1. INTRODUCTION

Let  $X$  be a Banach space and let  $L(X)$  denote the algebra of all bounded linear operators on  $X$ . A bounded linear operator  $T$  on  $X$  is *cyclic* if there exists a (cyclic) vector  $x$  such that the linear span of the orbit of  $x$ ,  $Orb(T, x) = \{T^n x : n = 0, 1, \dots\}$ , is dense in  $X$ . An operator  $T$  is called *convex-cyclic* if there exists a vector  $x \in X$  such that the convex hull of  $Orb(T, x)$  is dense in  $X$  and such a vector  $x$  is said to be a *convex-cyclic vector* for  $T$ . Clearly all convex-cyclic operators are cyclic. Following Rezaei [23] we will say that a polynomial  $p$  is a *convex polynomial* if it is a (finite) convex combination of monomials  $\{1, z, z^2, \dots\}$ . So,  $p(z) = a_0 + a_1 z + \dots + a_n z^n$  is a convex polynomial if  $a_k \geq 0$  for all  $k$  and  $\sum_{k=0}^n a_k = 1$ . Then the convex hull of an orbit is  $co(Orb(T, x)) = \{p(T)x : p \text{ is a convex polynomial}\}$ .

A bounded linear operator  $T \in L(X)$  is said to be *hypercyclic* (*weakly hypercyclic* [11]) if there is a vector  $x \in X$  whose orbit is dense in the norm (weak) topology of  $X$ . An operator  $T$  is said to be *weakly-mixing* if  $T \oplus T$  is hypercyclic in  $X \oplus X$ .

There are certainly examples of convex-cyclic operators that are not hypercyclic. However within certain classes of operators, hypercyclicity and convex-cyclicity are equivalent. This is true for unilateral weighted backward shifts on  $\ell^p(\mathbb{N})$  and composition operators on the classical Hardy space, see [23].

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What follows is a list of questions that are answered in this paper. First, notice that every weakly hypercyclic operator is convex-cyclic since the norm and the weak closure of a convex set in a Banach space coincide. In [23] Rezaei asks the following question:

**Question 1.** [23, Question 5.4] *Is every convex-cyclic operator acting on an infinite dimensional Banach space weakly hypercyclic?*

According to Feldman [16],  $T$  is called *1-weakly hypercyclic* if there is an  $x \in X$  such that  $f(\text{Orb}(T, x))$  is dense in  $\mathbb{C}$  for each non-zero  $f \in X^*$ . Every weakly hypercyclic operator is 1-weakly hypercyclic and 1-weakly hypercyclic operators are convex-cyclic. Thus it is also natural to ask if every convex-cyclic operator acting on an infinite dimensional Banach space is 1-weakly hypercyclic?

Ansari [2] showed that powers of hypercyclic operators on Banach spaces are hypercyclic operators. The same result was proven for operators on locally convex spaces by Bourdon and Feldman [10]. These results do not have analogues for cyclic operators. The forward unilateral shift  $S$  on  $\ell^2(\mathbb{N})$  is cyclic but  $S^2$  is not cyclic, because the codimension of the range of  $S^2$  is two. What about powers of convex-cyclic operators? León and Romero in [22] give examples of convex-cyclic operators where  $\sigma_p(S^*)$  is non-empty that have powers that are not convex-cyclic. Thus it is natural to ask:

**Question 2.** [23, Question 5.5] *If  $S : X \rightarrow X$  a convex-cyclic operator on a Banach space  $X$  with  $\sigma_p(S^*) = \emptyset$ , then is  $S^n$  convex-cyclic for every integer  $n > 1$ ?*

For a positive integer  $m$  and a positive real number  $p$ , an operator  $T \in L(X)$  is called an  $(m, p)$ -isometry if for any  $x \in X$ ,

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \|T^k x\|^p = 0.$$

An operator  $T$  is called an  $m$ -isometry if it is an  $(m, p)$ -isometry for some  $p > 0$ . See [1], [4] and [19]. Faghih and Hedayatian proved in [15] that  $m$ -isometries on a Hilbert space are not weakly hypercyclic. However, there are isometries that are weakly supercyclic [24] (in particular cyclic). Thus a natural question is the following:

**Question 3.** *Can an  $m$ -isometry be convex-cyclic?*

In [3], Badea, Grivaux and Müller introduced the concept of an  $\varepsilon$ -hypercyclic operator.

**Definition 1.1.** Let  $\varepsilon \in (0, 1)$  and let  $T : X \rightarrow X$  be a continuous linear operator. A vector  $x \in X$  is called an  $\varepsilon$ -hypercyclic vector for  $T$  if for every non-zero vector  $y \in X$  there exists

a non-negative integer  $n$  such that

$$\|T^n x - y\| \leq \varepsilon \|y\|.$$

The operator  $T$  is called  $\varepsilon$ -*hypercyclic* if it has an  $\varepsilon$ -hypercyclic vector.

In [3] it was shown that for every  $\varepsilon \in (0, 1)$ , there exists an  $\varepsilon$ -hypercyclic operator on the space  $\ell^1(\mathbb{N})$  which is not hypercyclic. Bayart in [5] extended this result to separable Hilbert spaces. Thus it is natural to ask if:

**Question 4.** *Is every  $\varepsilon$ -hypercyclic operator also convex-cyclic?*

An operator  $T \in L(X)$  is called *hypercyclic with support  $N$*  if there exists a vector  $x \in X$  such that the set

$$\{T^{k_1}x + T^{k_2}x + \cdots + T^{k_N}x \quad : \quad k_1, \dots, k_N \in \mathbb{N}\}$$

is dense in  $X$ .

**Remark 1.2.** Notice that if  $T$  is hypercyclic with support  $N$ , then  $T$  is convex-cyclic. In fact, for any  $y \in X$ , there exist  $k_1, \dots, k_N \in \mathbb{N}$  such that  $T^{k_1}x + T^{k_2}x + \cdots + T^{k_N}x \approx Ny$ , thus

$$\frac{T^{k_1}x + T^{k_2}x + \cdots + T^{k_N}x}{N} \approx y.$$

Any hypercyclic operator with support  $N$  satisfies that  $\sigma_p(T^*)$  is the empty set [6, Proposition 3.1]. However, there are convex-cyclic operators such that  $\sigma_p(T^*)$  is non-empty. So, hypercyclicity with support  $N$  is not equivalent to convex-cyclicity.

In [23], Rezaei characterizes which diagonal matrices on  $\mathbb{C}^n$  are convex-cyclic as those whose eigenvalues are distinct and belong to the set  $\mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$ . This naturally leads to the question about infinite diagonal matrices and even the following more general question.

**Question 5.** *If  $T$  is a continuous linear operator on a complex Banach space  $X$  and  $T$  has a complete set of eigenvectors whose eigenvalues are distinct, and belong to the set  $\mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$ , then is  $T$  convex-cyclic?*

In this paper, we answer these five questions and also give some examples. The paper is organized as follows. In Section 2, we give the Hahn-Banach characterization for convex-cyclicity. In Section 3, we give an example of an operator  $S$  that is convex-cyclic but  $S^2$  is not convex-cyclic and thus  $S$  is not weakly hypercyclic, this answers Questions 1 and 2 when  $\sigma_p(S^*) = \emptyset$ . In Section 4, we prove that  $m$ -isometries are not convex-cyclic, answering Question 3. In Section 5 we prove that any  $\varepsilon$ -hypercyclic operator is convex-cyclic. In fact, every  $\varepsilon$ -hypercyclic vector is a convex-cyclic vector. Finally, in Section 6 we answer Question

5 affirmatively and give examples of such operators including diagonal operators and adjoints of multiplication operators.

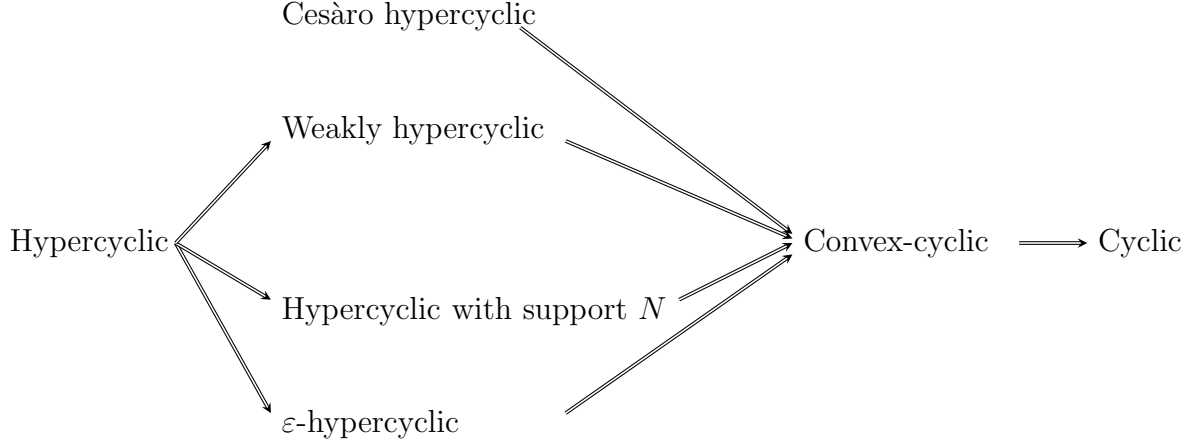


FIGURE 1. Implications between different definitions related with hypercyclicity and cyclicity.

## 2. THE HAHN-BANACH CHARACTERIZATION FOR CONVEX-CYCLICITY

Rezaei gave a (universality) criterion for an operator to be convex-cyclic [23, Theorem 3.10]. In the following result, using the Hahn-Banach Separation Theorem, we give a necessary and sufficient condition for a set to have a dense convex hull, as a result we get a criterion for a vector to be a convex-cyclic vector for an operator.

**Proposition 2.1.** *Let  $X$  be a locally convex space over the real or complex numbers and let  $E$  be a nonempty subset of  $X$ . The following are equivalent:*

- (1) *The convex hull of  $E$  is dense in  $X$ .*
- (2) *For every nonzero continuous linear functional  $f$  on  $X$  we have that the convex hull of  $\operatorname{Re}(f(E))$  is dense in  $\mathbb{R}$ .*
- (3) *For every nonzero continuous linear functional  $f$  on  $X$  we have that*

$$\sup \operatorname{Re}(f(E)) = \infty \text{ and } \inf \operatorname{Re}(f(E)) = -\infty.$$

- (4) *For every nonzero continuous linear functional  $f$  on  $X$  we have that*

$$\sup \operatorname{Re}(f(E)) = \infty.$$

*Proof.* Let  $\mathbb{F}$  denote either the real or complex numbers. Clearly  $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$  holds. Now assume that (4) holds and by way of contradiction, assume that  $\operatorname{co}(E)$  is not

dense in  $X$ . Then there exists a point  $p \in X$  that is not in the closure of  $\text{co}(E)$ . So, by the Hahn-Banach Separation Theorem ([12, Theorem 3.13]), there exists a continuous linear functional  $f$  on  $X$  so that  $\text{Re}(f(x)) < \text{Re}(f(p))$  for all  $x \in \text{co}(E)$ . It follows that  $\text{Re}(f(E))$  is bounded from above and thus  $\sup \text{Re}(f(E)) \neq \infty$ . This contradicts our assumption that (4) is true. Thus it must be the case that if (4) holds, then (1) does also. Hence all four conditions are equivalent.  $\square$

**Corollary 2.2** (The Hahn-Banach Characterization for Convex-Cyclicity). *Let  $X$  be a locally convex space over the real or complex numbers,  $T : X \rightarrow X$  a continuous linear operator, and  $x \in X$ . Then the following are equivalent:*

- (1) *The convex hull of the orbit of  $x$  under  $T$  is dense in  $X$ .*
- (2) *For every non-zero continuous linear functional  $f$  on  $X$  we have*

$$\sup \text{Re}(f(\text{Orb}(T, x))) = \infty.$$

Below are some simple consequences of the Hahn-Banach characterization for convex-cyclic vectors.

As it was pointed in the Introduction the range of a cyclic operator may not be dense. For example, the range of the unilateral shift has codimension one. However, the closure of the range of a cyclic operator has codimension at most one. Notice that the range of hypercyclic operator is always dense. The Hahn-Banach characterization of convex-cyclicity easily shows that convex-cyclic operators must also have dense range, see the following result.

**Proposition 2.3.** *If  $T$  is a convex-cyclic operator on a locally convex space  $X$ , then  $T$  has dense range.*

*Proof.* Suppose that  $T$  is a convex-cyclic operator and let  $x$  be a convex-cyclic vector for  $T$ , and by way of contradiction, suppose that  $T$  does not have dense range. Then there exists a continuous linear functional  $f$  such that  $f(R(T)) = \{0\}$ , where  $R(T)$  denotes the range of  $T$ . By the Hahn-Banach characterization, Corollary 2.2, we must have that  $\sup \text{Re}(f(\text{Orb}(T, x))) = \infty$ . However, since  $T^n x \in R(T)$  for all  $n \geq 1$  it follows that  $f(T^n x) = 0$  for all  $n \geq 1$ . So,  $\sup \text{Re}(f(\text{Orb}(T, x))) = \sup \text{Re}(\{f(T^0 x), 0\}) < \infty$ . It follows from Corollary 2.2 that  $x$  is not a convex-cyclic vector, a contradiction. Thus,  $T$  must have dense range.  $\square$

In general, if  $T$  is hypercyclic and  $c > 1$ , then  $cT$  may not be hypercyclic. However, León-Saavedra and Müller [21] proved that if  $T$  is hypercyclic and  $\alpha$  is a unimodular complex number, then  $\alpha T$  is hypercyclic. The same property is also true for weak hypercyclic operators [13, Theorem 2.8]. Next we present a similar result for convex-cyclic operators, that follows from the Hahn-Banach characterization of convex-cyclic vectors.

**Proposition 2.4.** *If  $T$  is a convex-cyclic operator on a real or complex locally convex space  $X$ , and if  $c > 1$ , then  $cT$  is also convex-cyclic. Furthermore, every convex-cyclic vector for  $T$  is also a convex-cyclic vector for  $cT$ .*

*Proof.* Suppose that  $x$  is a convex-cyclic vector for  $T$ , and we will show that  $x$  is also a convex-cyclic vector for  $cT$ , by using the Hahn-Banach characterization (Corollary 2.2). Let  $f$  be any non-zero continuous linear functional on  $X$ . Since  $x$  is a convex-cyclic vector for  $T$ , then  $\sup \operatorname{Re}(f(T^n x)) = \infty$ . Since  $c > 1$ , then we have that  $\sup \operatorname{Re}[f((cT)^n x)] = \sup c^n \operatorname{Re}[f(T^n x)] \geq \sup \operatorname{Re}[f(T^n x)] = \infty$ . So, by the Hahn-Banach characterization,  $x$  is a convex-cyclic vector for  $cT$ .  $\square$

**Corollary 2.5.** *If  $|c| \geq 1$  and  $T$  is weakly hypercyclic, then  $cT$  is convex-cyclic.*

*Proof.* Let  $c := e^{i\theta}\beta$ , where  $\theta \in \mathbb{R}$  and  $\beta \geq 1$ . Then by de la Rosa [13, Theorem 2.8] we obtain that  $e^{i\theta}T$  is weakly hypercyclic, hence  $e^{i\theta}T$  is convex-cyclic. Thus,  $cT = \beta(e^{i\theta}T)$  is convex cyclic by Proposition 2.4.  $\square$

Let us define the following convex polynomials

$$p_k^c(t) := \begin{cases} \frac{1+t+\dots+t^{k-1}}{k} & \text{if } c = 1 \\ \frac{c-1}{c^k-1}(c^{k-1}+c^{k-2}t+\dots+t^{k-1}) & \text{if } c > 1. \end{cases}$$

**Definition 2.6.** Let  $X$  and  $Y$  be topological spaces. A family of continuous operators  $T_i : X \rightarrow Y$  ( $i \in I$ ) is *universal* if there exists an  $x \in X$  such that  $\{T_i x : i \in I\}$  is dense in  $Y$ .

Let  $T \in L(X)$ . Denotes  $M_n(T)$  the *arithmetic means* given by

$$M_n(T) := \frac{I + T + \dots + T^{n-1}}{n}.$$

Recall that an operator  $T$  is *Cesàro hypercyclic* if there exists  $x \in X$  such that  $\{M_n(T)x : n \in \mathbb{N}\}$  is dense in  $X$ . See [20].

In [20, Theorem 2.4] it is proved that  $T$  is Cesàro hypercyclic if and only if  $\left(\frac{T^k}{k}\right)_{k=1}^{\infty}$  is universal.

**Proposition 2.7.** *Let  $X$  be a Banach space,  $c > 1$  and  $T \in L(X)$  such that  $cI - T$  has dense range. Then the following are equivalent:*

- (1)  $\frac{T}{c}$  is hypercyclic
- (2)  $(p_k^c(T))_{k \in \mathbb{N}}$  is universal.

*Proof.* Notice that if  $c > 1$ ,

$$p_k^c(T)(cI - T)x = (cI - T)p_k^c(T)x = (c - 1)\frac{c^k}{c^k - 1} \left( x - \left( \frac{T}{c} \right)^k x \right).$$

□

**Proposition 2.8.** *If  $T$  is Cesàro hypercyclic or  $\frac{T}{c}$  is hypercyclic for some  $c \geq 1$ , then  $T$  is convex-cyclic.*

Notice that the proof of the sufficient condition for a bilateral weighted backward shift on  $\ell^p(\mathbb{Z})$  to be convex-cyclic given in [23, Theorem 4.2] is not correct.

### 3. CONVEX-CYCLIC OPERATORS WHOSE SQUARES ARE NOT CONVEX-CYCLIC

As noted in the Introduction, powers of hypercyclic and weakly hypercyclic operators remain hypercyclic and weakly hypercyclic, respectively. In this section, we give an example of a convex-cyclic operator  $S$  with  $\sigma_p(S^*) = \emptyset$  such that  $S^2$  is not convex-cyclic. Moreover, the same example gives an operator that is convex-cyclic with  $\sigma_p(S^*) = \emptyset$  that is not weakly hypercyclic.

Recall that León-Saavedra and Romero de la Rosa [22] provide an example of a convex-cyclic operator  $S$  with  $\sigma_p(S^*) \neq \emptyset$  such that  $S^n$  fails to be convex-cyclic. Also, a  $2 \times 2$  diagonal matrix  $D$  with eigenvalues  $2i$  and  $-2i$  is convex-cyclic, but  $D^2$  has a real eigenvalue and thus is not convex-cyclic.

**Theorem 3.1.** [17] *Let  $T$  be a hypercyclic operator on an infinite dimensional separable Banach space. The following assertions are equivalent:*

- (1)  $T \oplus T$  is hypercyclic.
- (2)  $T \oplus T$  is cyclic.

**Theorem 3.2.** ([6, Proposition 2.3] & [28, Corollary 5.2]) *Let  $T$  be a hypercyclic operator on a separable Banach space. Then  $T \oplus -T$  is hypercyclic with support 2 and 1-weakly hypercyclic.*

**Corollary 3.1.** *If  $T$  is a hypercyclic operator on an infinite dimensional Banach space such that  $T \oplus T$  is not hypercyclic, then  $T \oplus -T$  is convex-cyclic, but not weakly hypercyclic and  $(T \oplus -T)^2$  is not cyclic.*

*Proof.* Suppose that  $T$  is a hypercyclic operator such that  $T \oplus T$  is not hypercyclic. Then by Theorem 3.1,  $T \oplus T$  is not cyclic. Thus  $(T \oplus T)^2$  is not cyclic, hence  $(T \oplus -T)^2 = (T \oplus T)^2$  is not cyclic. It follows that  $T \oplus -T$  is not weakly hypercyclic, for if it was, then  $(T \oplus -T)^2 =$

$(T \oplus T)^2$  would be weakly hypercyclic, and hence cyclic, a contradiction. Thus  $T \oplus -T$  is convex-cyclic but not weakly hypercyclic, and  $(T \oplus -T)^2$  is not cyclic.  $\square$

Examples of operators satisfying that  $T$  is hypercyclic but  $T \oplus T$  is not hypercyclic are given in [14], [8, Corollary 4.15] and [7]. Using these examples we have the following result.

**Theorem 3.3.** *There exists an operator  $S$  on  $c_0(\mathbb{N}) \oplus c_0(\mathbb{N})$  or on  $\ell^p(\mathbb{N}) \oplus \ell^p(\mathbb{N})$  with  $p \geq 1$  that is convex-cyclic, but not weakly hypercyclic, and  $S^2$  is not convex-cyclic.*

Using similar ideas of Shkarin [27, Lemma 6.5] we obtain the following result.

**Theorem 3.4.** *Let  $T \in L(X)$ . If  $T^2$  is convex-cyclic, then  $T \oplus -T$  is convex-cyclic.*

*Proof.* Let  $x$  be a convex-cyclic vector for  $T^2$  and let  $S := T \oplus -T$ . Then for all  $y \in X$  there exists a sequence  $(p_k)$  of convex polynomials such that  $p_k(T^2)x$  converges to  $y$  as  $k$  tends to infinity. Thus

$$p_k(S^2)(x, x) \rightarrow (y, y),$$

and

$$Sp_k(S^2)(x, x) \rightarrow (Ty, -Ty).$$

Since  $T^2$  is convex-cyclic,  $T$  is convex-cyclic. By Proposition 2.3, the range of  $T$  is dense. By other hand,  $p_k(x^2)$  and  $xp_k(x^2)$  are convex polynomials. Thus the closed convex hull of  $\text{Orb}(S, (x, x))$  contains the spaces  $L_0 := \{(u, u) : u \in X\}$  and  $L_1 := \{(u, -u) : u \in X\}$ . So, if we are given  $(y, z) \in X \times X$ , then let  $(q_k)$  and  $(h_k)$  be sequences of convex polynomials such that

$$q_k(S^2)(x, x) \rightarrow (y + z, y + z)$$

and

$$Sh_k(S^2)(x, x) \rightarrow (y - z, z - y).$$

Then  $p_k(t) := \frac{1}{2}q_k(t^2) + \frac{t}{2}h_k(t^2)$  is a sequence of convex polynomials and

$$p_k(S)(x, x) \rightarrow (y, z).$$

Thus  $S = T \oplus -T$  is convex-cyclic.  $\square$

Ansari [2] proved that an operator  $T$  is hypercyclic if and only if  $T^n$  is hypercyclic. In fact  $T$  and  $T^n$  have the same set of hypercyclic vectors for any positive integer  $n$ . This property is also true for weakly hypercyclic vectors (see [10, Theorem 2.4]), thus we get the following corollary.

**Corollary 3.2.** *If  $T$  is weakly-hypercyclic, then  $T \oplus -T$  is convex-cyclic.*



In the following result we obtain that if  $T$  and  $T^n$  are convex-cyclic operators, the set of convex-cyclic vectors could be different.

**Proposition 3.3.** *There are hypercyclic operators such that  $T$  and  $T^2$  do not have the same convex-cyclic vectors.*

*Proof.* Let  $T$  be twice the backward shift,  $T := 2B$ , on  $\ell^2(\mathbb{N})$  and let  $D$  be the doubling map on  $\ell^2(\mathbb{N})$ , given by  $D(x_0, x_1, x_2, \dots) = (x_0, x_0, x_1, x_1, x_2, x_2, \dots)$ . By [16, Theorem 5.3] there exists an  $x \in \ell^2(\mathbb{N})$  such that  $x$  is a 1-weakly hypercyclic vector for  $T$  (and hence a convex-cyclic vector for  $T$ ) and  $\text{Orb}(T^2, x) \subseteq D(\ell^2(\mathbb{N}))$ . Thus,

$$\overline{\text{co}(\text{Orb}(T^2, x))} \subseteq \overline{\text{span}[\text{Orb}(T^2, x)]} \subseteq D(\ell^2(\mathbb{N})) \neq \ell^2(\mathbb{N}).$$

Since  $D(\ell^2(\mathbb{N}))$  is a proper closed subspace of  $\ell^2(\mathbb{N})$ , this complete the proof.  $\square$

#### 4. M-ISOMETRIES ARE NOT CONVEX-CYCLIC

Bayart proved the following spectral result for  $m$ -isometries on Banach spaces.

**Proposition 4.1.** [4, Proposition 2.3] *Let  $T \in L(X)$  be an  $m$ -isometry. Then its approximate point spectrum lies in the unit circle. In particular,  $T$  is one-to-one,  $T$  has closed range and either  $\sigma(T) \subseteq \mathbb{T}$  or  $\sigma(T) = \overline{\mathbb{D}}$ .*

On the other hand, Rezaei proved the following properties for convex-cyclic operators.

**Proposition 4.2.** [23, Propositions 3.2 and 3.3] *Let  $T \in L(X)$ . If  $T$  is convex-cyclic, then*

- (1)  $\|T\| > 1$ .
- (2)  $\sigma_p(T^*) \subset \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$ .

**Theorem 4.1.** *An  $m$ -isometry on a Banach space  $X$  is not convex-cyclic.*

*Proof.* If  $m = 1$  and  $T$  is an  $m$ -isometry, then  $T$  is actually an isometry, thus  $\|T\| = 1$  and thus by part (1) of Proposition 4.2,  $T$  cannot be convex-cyclic.

Assume that  $m \geq 2$  and that  $T$  is convex-cyclic and a strict  $(m, p)$ -isometry for some  $p > 0$ . We will use an argument similar to the proof of [4, Theorem 3.3]. Let

$$|x| := \lim_{n \rightarrow \infty} \frac{\|T^n x\|}{n^{\frac{m-1}{p}}}.$$

By [4, Proposition 2.2] we have that  $|\cdot|$  is a semi-norm on  $X$  and  $T(\text{Ker}(|\cdot|)) \subset \text{Ker}(|\cdot|)$ , where  $\text{Ker}(T)$  denotes the kernel of  $T$ . Also the codimension of  $\text{Ker}(|\cdot|)$  is positive, because  $T$  is not a  $(m-1)$ -isometry. Moreover, for each  $x \in X$ ,  $|Tx| = |x|$  and there exists  $C > 0$  such that  $|x| \leq C\|x\|$  for all  $x \in X$ .

Let  $Y := X/\text{Ker}(|\cdot|)$  and  $\bar{T}$  be the operator induced by  $T$  on  $Y$ . Then  $|\bar{T}\bar{x}| = |\bar{x}|$  for all  $\bar{x} \in Y$ . So,  $\bar{T}$  is an isometry on  $Y$ .

Since  $T$  is convex-cyclic there exists a vector  $x \in X$  such that the convex hull generated by  $\text{Orb}(T, x)$  is dense in  $X$ . Given  $y \in X$  and  $\varepsilon > 0$  there exists a convex polynomial such that  $\|y - p_n(T)x\| < \frac{\varepsilon}{C}$ . Thus  $|y - p_n(T)x| \leq C\|y - p_n(T)x\| < \varepsilon$ . Then  $|\bar{y} - p_n(\bar{T})\bar{x}| < \varepsilon$  and we obtain that  $\bar{T}$  is convex-cyclic in  $Y$ .

Thus the extension of  $\bar{T}$  to the completion of  $Y$  is a convex-cyclic isometry on a Banach space; which is a contradiction.  $\square$

**Corollary 4.3.** *An  $m$ -isometry on a Banach space is not 1-weakly hypercyclic.*

## 5. $\varepsilon$ -HYPERCYCLIC OPERATORS VERSUS CONVEX-CYCLIC OPERATORS

Let us now exhibit the relation between  $\varepsilon$ -hypercyclic and convex-cyclic operators.

**Theorem 5.1.** *Every  $\varepsilon$ -hypercyclic vector is a convex-cyclic vector.*

*Proof.* Let  $x$  be an  $\varepsilon$ -hypercyclic vector for an operator  $T$  and we will prove that for a non-zero vector  $y \in X$  and  $\delta > 0$ , there exists a convex polynomial  $p$  such that

$$\|p(T)x - y\| < \delta.$$

Since  $\varepsilon \in (0, 1)$ , there exists  $N \in \mathbb{N}$  such that  $2\varepsilon^N\|y\| < \delta$ . As  $x$  is an  $\varepsilon$ -hypercyclic vector for  $T$ , there exists a positive integer  $k_1$  such that

$$\|T^{k_1}x - Ny\| \leq \varepsilon\|Ny\| = \varepsilon N\|y\|.$$

If  $T^{k_1}x - Ny = 0$ , we choose  $l_2$  such that

$$\left\| T^{l_2}x - \frac{N}{N-1}\varepsilon^N y \right\| \leq \varepsilon^{N+1} \frac{N}{N-1} \|y\|.$$

Thus

$$\left\| \frac{N-1}{N} T^{l_2}x - \varepsilon^N y \right\| \leq \varepsilon^{N+1} \|y\|.$$

Hence

$$\left\| \frac{1}{N} T^{k_1}x + \frac{N-1}{N} T^{l_2}x - y \right\| = \left\| \frac{N-1}{N} T^{l_2}x \right\| \leq 2\varepsilon^N \|y\| < \delta$$

and the proof ends by letting  $p(z) = \frac{1}{N}z^{k_1} + \frac{N-1}{N}z^{l_2}$ .

If  $T^{k_1}x - Ny \neq 0$ , there exists a positive integer  $k_2$  such that

$$\|T^{k_1}x + T^{k_2}x - Ny\| = \|T^{k_2}x - (Ny - T^{k_1}x)\| \leq \varepsilon\|Ny - T^{k_1}x\| \leq \varepsilon^2 N\|y\|.$$

If  $T^{k_1}x + T^{k_2}x - Ny = 0$ , analogously to the above situation we choose  $l_3$  such that

$$\left\| \frac{1}{N}T^{k_1}x + \frac{1}{N}T^{k_2}x + \frac{N-2}{N}T^{l_3}x - y \right\| = \left\| \frac{N-2}{N}T^{l_3}x \right\| \leq 2\varepsilon^N\|y\| < \delta$$

and the proof ends.

If  $T^{k_1}x + T^{k_2}x - Ny \neq 0$ , there exists a positive integer  $k_3$  such that

$$\|T^{k_1}x + T^{k_2}x + T^{k_3}x - Ny\| \leq \varepsilon^3 N\|y\|.$$

By induction, in the step  $N$ , if  $T^{k_1}x + T^{k_2}x + \dots + T^{k_{N-1}}x - Ny = 0$ , we choose  $l_N$  such that

$$\left\| \frac{1}{N}T^{k_1}x + \frac{1}{N}T^{k_2}x + \dots + \frac{1}{N}T^{k_{N-1}}x + \frac{1}{N}T^{l_N}x - y \right\| \leq 2\varepsilon^N\|y\| < \delta$$

and the proof ends.

If  $T^{k_1}x + T^{k_2}x + \dots + T^{k_{N-1}}x - Ny \neq 0$ , there exists a positive integer  $k_N$  such that

$$\|T^{k_1}x + T^{k_2}x + \dots + T^{k_{N-1}}x + T^{k_N}x - Ny\| \leq \varepsilon^N N\|y\|$$

Thus

$$\left\| \frac{T^{k_1}x + \dots + T^{k_N}x}{N} - y \right\| \leq \varepsilon^N\|y\| < \delta$$

Ending completely the proof.  $\square$

## 6. DIAGONAL OPERATORS AND ADJOINT MULTIPLICATION OPERATORS

By a Fréchet space we mean a locally convex space that is complete with respect to a translation invariant metric.

If  $\mathcal{A}$  is a nonempty collection of polynomials and  $T$  is an operator on a space  $X$ , then  $T$  is said to be  $\mathcal{A}$ -cyclic and  $x \in X$  is said to be an  $\mathcal{A}$ -cyclic vector for  $T$  if  $\{p(T)x : p \in \mathcal{A}\}$  is dense in  $X$ . Furthermore,  $T$  is said to be  $\mathcal{A}$ -transitive if for any two nonempty open sets  $U$  and  $V$  in  $X$ , there exists a  $p \in \mathcal{A}$  such that  $p(T)U \cap V \neq \emptyset$ . Since the set of all polynomials with the topology of uniform convergence on compact sets in the complex plane forms a separable metric space, then any set of polynomials is also separable, hence the following result is routine (see for example the Universality Criterion in [18, Theorem 1.57]).

**Proposition 6.1.** *Suppose that  $T : X \rightarrow X$  is a continuous linear operator on a real or complex Fréchet space and  $\mathcal{A}$  is a nonempty set of polynomials. Then the following are equivalent:*

- (1)  $T$  has a dense set of  $\mathcal{A}$ -cyclic vectors.
- (2)  $T$  is  $\mathcal{A}$ -transitive. That is, for any two nonempty open sets  $U, V$  in  $X$ , there is a polynomial  $p \in \mathcal{A}$  such that  $p(T)U \cap V \neq \emptyset$ .
- (3)  $T$  has a dense  $G_\delta$  set of  $\mathcal{A}$ -cyclic vectors.

By choosing various sets of polynomials for  $\mathcal{A}$ , we can get results for hypercyclic and supercyclic operators, as well as cyclic operators that have a dense set of cyclic vectors. If  $\mathcal{A}$  is the set of all convex polynomials, then we get the following immediate corollary.

**Corollary 6.2.** *Let  $T : X \rightarrow X$  be a continuous linear operator on a real or complex Fréchet space, then the following are equivalent.*

- (1)  *$T$  has a dense set of convex-cyclic vectors.*
- (2)  *$T$  is convex-transitive. That is, for any two nonempty open sets  $U, V$  in  $X$ , there is a convex polynomial  $p$  such that  $p(T)U \cap V \neq \emptyset$ .*
- (3)  *$T$  has a dense  $G_\delta$  set of convex-cyclic vectors.*

**Proposition 6.3.** *Let  $\mathcal{A}$  be a nonempty set of polynomials and let  $\{T_k : X_k \rightarrow X_k\}_{k=1}^\infty$  be a uniformly bounded sequence of linear operators on a sequence of Banach spaces  $\{X_k\}_{k=1}^\infty$  such that for every  $n \geq 1$ , the operator  $S_n = \bigoplus_{k=1}^n T_k$  on  $X^{(n)} = \bigoplus_{k=1}^n X_k$  has a dense set of  $\mathcal{A}$ -cyclic vectors. Then  $T = \bigoplus_{k=1}^\infty T_k$  is  $\mathcal{A}$ -cyclic on  $X^{(\infty)} = \bigoplus_{k=1}^\infty X_k$  and  $T$  has a dense set of  $\mathcal{A}$ -cyclic vectors.*

*Proof.* Suppose that for every  $n \geq 1$  the operators  $S_n$  are  $\mathcal{A}$ -cyclic and have a dense set of  $\mathcal{A}$ -cyclic vectors. We will show that  $T$  is  $\mathcal{A}$ -transitive. Let  $U$  and  $V$  be two nonempty open sets in  $X^{(\infty)}$ . Since the vectors in  $X$  with only finitely many non-zero coordinates are dense in  $X$ , then we may choose vectors  $x = (x_k)_{k=1}^\infty$  and  $y = (y_k)_{k=1}^\infty$  in  $X^{(\infty)}$  such that  $x_k = 0$  and  $y_k = 0$  for all sufficiently large  $k$ , say  $x_k = 0$  and  $y_k = 0$  for all  $k \geq N$ , and such that  $x \in U$  and  $y \in V$ . Since  $S_N$  is  $\mathcal{A}$ -cyclic and has a dense set of  $\mathcal{A}$ -cyclic vectors in  $X^{(N)}$ , there exists a vector  $u = (u_1, u_2, \dots, u_N) \in X^{(N)}$  such that  $u$  is an  $\mathcal{A}$ -cyclic vector for  $S_N$  and so that  $(u_1, u_2, \dots, u_N)$  is close enough to  $(x_1, x_2, \dots, x_N)$  so that the infinite vector  $\hat{u} = (u_1, u_2, \dots, u_N, 0, 0, \dots) \in U$ . Since  $S_N$  is  $\mathcal{A}$ -cyclic, there is a polynomial  $p \in \mathcal{A}$  such that  $p(S_N)(u_1, u_2, \dots, u_N)$  is close enough to  $(y_1, y_2, \dots, y_N)$  such that  $p(T)\hat{u} \in V$ . Thus,  $T$  is  $\mathcal{A}$ -transitive on  $X^{(\infty)}$ , and thus by Proposition 6.1 we have that  $T$  has a dense set of  $\mathcal{A}$ -cyclic vectors.  $\square$

We next apply the previous proposition to infinite diagonal operators where  $\mathcal{A}$  is the set of all convex polynomials. This extends the finite dimensional matrix result given by Rezaei [23, Corollary 2.7] to infinite dimensional diagonal matrices.

**Theorem 6.1.** *Suppose that  $T$  is a diagonalizable normal operator on a separable (real or complex) Hilbert space with eigenvalues  $\{\lambda_k\}_{k=1}^\infty$ .*

(a) If the Hilbert space is complex, then  $T$  is convex-cyclic if and only if we have that the eigenvalues  $\{\lambda_k\}_{k=1}^\infty$  are distinct and for every  $k \geq 1$ ,  $|\lambda_k| > 1$  and  $\text{Im}(\lambda_k) \neq 0$ .

(b) If the Hilbert space is real, then  $T$  is convex-cyclic if and only if the eigenvalues  $\{\lambda_k\}_{k=1}^\infty$  are distinct and for every  $k \geq 1$  we have that  $\lambda_k < -1$ .

*Proof.* By the spectral theorem we may assume that  $T = \text{diag}(\lambda_1, \lambda_2, \dots)$  is an infinite diagonal matrix acting on  $\ell_{\mathbb{C}}^2(\mathbb{N})$  and let  $\{e_k\}_{k=1}^\infty$  be the canonical unit vector basis where  $e_k$  has a one in its  $k^{\text{th}}$  coordinate and zeros elsewhere.

(a) If  $T$  is convex-cyclic with convex-cyclic vector  $x = (x_n)_{n=1}^\infty \in \ell_{\mathbb{C}}^2(\mathbb{N})$ , then by Corollary 2.2 we must have for every  $k \geq 1$  that  $\infty = \sup_{n \geq 1} \text{Re}(\langle T^n x, e_k \rangle) = \sup_{n \geq 1} \text{Re}(\lambda_k^n x_k)$ . This implies that  $x_k \neq 0$  and that  $|\lambda_k| > 1$  for each  $k \geq 1$ . Likewise, since the Hilbert space is complex in this case, we must have

$$\infty = \sup_{n \geq 1} \text{Re} \left( \langle T^n x, \frac{-i}{x_k} e_k \rangle \right) = \sup_{n \geq 1} \text{Re} \left( \lambda_k^n x_k \frac{i}{x_k} \right) = \sup_{n \geq 1} \text{Re}(i \lambda_k^n).$$

This implies that  $\lambda_k$  cannot be real, hence  $\text{Im}(\lambda_k) \neq 0$  for all  $k \geq 1$ .

Conversely, suppose that for every  $k \geq 1$  we have that  $|\lambda_k| > 1$  and  $\text{Im}(\lambda_k) \neq 0$ . Then for  $n \geq 1$ , let  $T_n := \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$  be the diagonal matrix on  $\mathbb{C}^n$  where  $\lambda_k$  is the  $k^{\text{th}}$  diagonal entry. Since the eigenvalues  $\{\lambda_k\}_{k=1}^\infty$  are distinct and  $|\lambda_k| > 1$  and  $\text{Im}(\lambda_k) \neq 0$  for  $1 \leq k \leq n$ , then we know from Rezaei [23] that  $T_n$  is convex-cyclic on  $\mathbb{C}^n$  and that every vector all of whose coordinates are non-zero is a convex-cyclic vector for  $T_n$ . Since such vectors are dense in  $\mathbb{C}^n$  for every  $n \geq 1$ , then it follows from Proposition 6.3 that  $T$  is also convex-cyclic and has a dense set of convex-cyclic vectors. (b) The proof of the real case is similar to that above.  $\square$

The next theorem says that if an operator has a complete set of eigenvectors whose eigenvalues are distinct, not real, and lie outside of the closed unit disk, then the operator is convex-cyclic.

**Theorem 6.2.** *Let  $S := \{re^{i\theta} : r > 1 \text{ and } 0 < |\theta| < \pi\} = \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$ . Suppose that  $T$  is a bounded linear operator on a complex Banach space  $X$  and that  $T$  has a countable linearly independent set of eigenvectors with dense linear span in  $X$  such that the corresponding eigenvalues are distinct and are contained in the set  $S$ . Then  $T$  is convex-cyclic and has a dense set of convex-cyclic vectors.*

*Proof.* Suppose that  $\{v_n\}_{n=1}^\infty$  is a linearly independent set of eigenvectors for  $T$  that have dense linear span in  $X$  and such that the corresponding eigenvalues  $\{\lambda_n\}_{n=1}^\infty$  are distinct and contained in the set  $S$ . By replacing each eigenvector  $v_n$  with a constant multiple of itself we may assume that  $\sum_{n=1}^\infty \|v_n\|^2 < \infty$ . Let  $D$  be the diagonal normal matrix on  $\ell^2(\mathbb{N})$  whose  $n^{\text{th}}$

diagonal entry is  $\lambda_n$ . Then define a linear map  $A : \ell^2(\mathbb{N}) \rightarrow X$  by  $A(\{a_n\}_{n=1}^\infty) = \sum_{n=1}^\infty a_n v_n$ . Notice that since  $\{a_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$ , then we have that

$$\|A(\{a_n\}_{n=1}^\infty)\| = \left\| \sum_{n=1}^\infty a_n v_n \right\| \leq \left( \sum_{n=1}^\infty |a_n|^2 \right)^{1/2} \left( \sum_{n=1}^\infty \|v_n\|^2 \right)^{1/2} = C \|\{a_n\}_{n=1}^\infty\|_{\ell^2(\mathbb{N})}$$

where  $C := (\sum_{n=1}^\infty \|v_n\|^2)^{1/2}$ , which is finite. The above inequality implies that  $A$  is a well defined continuous linear map from  $\ell^2(\mathbb{N})$  to  $X$ . It follows that since the eigenvectors  $\{v_n\}_{n=1}^\infty$  have dense linear span in  $X$ , that  $A$  has dense range. Also, if  $\{e_n\}_{n=1}^\infty$  is the standard unit vector basis in  $\ell^2(\mathbb{N})$ , then clearly  $A(e_n) = v_n$  for all  $n \geq 1$  and thus  $A$  intertwines  $D$  with  $T$ . To see this notice that  $AD(e_n) = A(\lambda_n e_n) = \lambda_n v_n = T(v_n) = TA(e_n)$ . Thus  $AD(e_n) = TA(e_n)$  for all  $n \geq 1$ , thus  $AD = TA$ . Finally, since  $D$  has distinct eigenvalues that all lie in the set  $S$ , it follows from Proposition 6.1 that  $D$  is convex-cyclic and has a dense set of convex-cyclic vectors. Since  $A$  intertwines  $D$  and  $T$  and  $A$  has dense range, then  $A$  will map convex-cyclic vectors for  $D$  to convex-cyclic vectors for  $T$ . Thus,  $T$  is convex-cyclic and has a dense set of convex-cyclic vectors.  $\square$

If  $G$  is an open set in the complex plane, then by a reproducing kernel Hilbert space  $\mathcal{H}$  of analytic functions on  $G$  we mean a vector space of analytic functions on  $G$  that is complete with respect to a norm given by an inner product and such that point evaluations at all points in  $G$  are continuous linear functionals on  $\mathcal{H}$ . Naturally we also require that  $f = 0$  in  $\mathcal{H}$  if and only if  $f(z) = 0$  for all  $z \in G$ . This is equivalent to the reproducing kernels having dense linear span in  $\mathcal{H}$ . Given such a space  $\mathcal{H}$ , a multiplier of  $\mathcal{H}$  is an analytic function  $\varphi$  on  $G$  so that  $\varphi f \in \mathcal{H}$  for every  $f \in \mathcal{H}$ . In this case, the closed graph theorem implies that the multiplication operator  $M_\varphi : \mathcal{H} \rightarrow \mathcal{H}$  is a bounded linear operator.

**Corollary 6.4.** *Suppose that  $G$  is an open set in  $\mathbb{C}$  with components  $\{G_n\}_{n \in J}$  and  $\mathcal{H}$  is a reproducing kernel Hilbert space of analytic functions on  $G$ , and that  $\varphi$  is a multiplier of  $\mathcal{H}$ . If  $\varphi$  is non-constant on every component of  $G$  and  $\varphi(G_n) \cap \{z \in \mathbb{C} : |z| > 1\} \neq \emptyset$  for every  $n \in J$ , then the operator  $M_\varphi^*$  is convex-cyclic on  $\mathcal{H}$  and has a dense set of convex-cyclic vectors.*

*Proof.* We will show that the eigenvectors for  $M_\varphi^*$  with eigenvalues in the set  $S = \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$  have dense linear span in  $\mathcal{H}$ . It will then follow from Theorem 6.2 that  $M_\varphi^*$  is convex-cyclic.

Every reproducing kernel for  $\mathcal{H}$  is an eigenvector for  $M_\varphi^*$ . In fact, if  $\lambda \in G$ , then  $M_\varphi^* K_\lambda = \overline{\varphi(\lambda)} K_\lambda$ , where  $K_\lambda$  denotes the reproducing kernel for  $\mathcal{H}$  at the point  $\lambda \in G$ . By assumption, for every component  $G_n$  of  $G$ ,  $\varphi$  is non-constant on  $G_n$ , thus the set  $\{\lambda \in G_n : |\varphi(\lambda)| > 1\}$  is a nonempty open subset of  $G_n$ . Also since  $\varphi$  is an open map on  $G_n$ ,  $\varphi$  cannot map the open set  $\{\lambda \in G_n : |\varphi(\lambda)| > 1\}$  into  $\mathbb{R}$ . Thus, for all  $n \in J$ ,  $E_n = \{\lambda \in G_n : |\varphi(\lambda)| > 1 \text{ and } \varphi(\lambda) \notin \mathbb{R}\}$

is a nonempty open subset of  $G_n$ . Let  $E := \bigcup_{n \in J} E_n$ . Then for every  $\lambda \in E$ ,  $K_\lambda$  is an eigenvector for  $M_\varphi^*$  with eigenvalue  $\overline{\varphi(\lambda)}$  which lies in  $S = \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$ . Since  $E \cap G_n$  is a nonempty open set for every  $n \in J$ , then the corresponding reproducing kernels  $\{K_\lambda : \lambda \in E\}$  have dense linear span in  $\mathcal{H}$ . Finally, since  $\varphi$  is non-constant on  $E_n$  for each  $n \in J$ , we can choose a countable set  $\{\lambda_{n,k}\}_{k=1}^\infty$  in  $E_n$  that has an accumulation point in  $E_n$  in such a way that  $\varphi$  is one-to-one on  $\{\lambda_{n,k}\}_{n,k=1}^\infty$ . Then the countable set  $\{K_{\lambda_{n,k}}\}_{n,k=1}^\infty$  is a set of independent eigenvectors with dense linear span in  $\mathcal{H}$  and with distinct eigenvalues. It now follows from Theorem 6.2 that  $M_\varphi^*$  is convex-cyclic and has a dense set of convex-cyclic vectors.  $\square$

**Remark 6.5.** *In the previous corollary, if  $G$  is an open connected set,  $\varphi$  is a non-constant-multiplier of  $\mathcal{H}$  and if the norm of  $M_\varphi$  is equal to its spectral radius, then  $M_\varphi^*$  is convex-cyclic if and only if  $\varphi(G) \cap \{z \in \mathbb{C} : |z| > 1\} \neq \emptyset$ . This is the case if  $\mathcal{H}$  is equal to  $H^2(G)$  or  $L_a^2(G)$ , the Hardy space or Bergman space on  $G$  or if  $M_\varphi$  is hyponormal.*

Next we give an example of a convex-cyclic operator that is not 1-weakly hypercyclic.

**Example 6.6.** Let  $M_{2+z}^*$  be the adjoint of the multiplication operator associated to the multiplier  $\varphi(z) := 2+z$  on  $H^2(\mathbb{D})$ . By [28, Theorem 5.5] the operator  $M_{2+z}^* = 2I + B$ , where  $B$  is the unilateral backward shift, is not 1-weakly-hypercyclic, however  $M_{2+z}^*$  is convex-cyclic by Corollary 6.4.

The following result is true since powers of convex polynomials are also convex polynomials.

**Proposition 6.7.** *If  $T$  is an operator on a Banach space and there exists a convex polynomial  $p$  such that  $p(T)$  is hypercyclic, then  $T$  is convex-cyclic.*

By a *region* in  $\mathbb{C}$  we mean an open connected set in  $\mathbb{C}$ . In the following theorem, we consider the operator which is the adjoint of multiplication by  $z$ , the independent variable.

**Theorem 6.3.** *Suppose that  $G$  is a bounded region in  $\mathbb{C}$  and  $G \cap \{z : |z| > 1\} \neq \emptyset$ . Suppose also that  $\mathcal{H}$  is a reproducing kernel Hilbert space of analytic functions on  $G$ , then  $M_z^*$  is convex-cyclic on  $\mathcal{H}$ . In fact, there exists a convex polynomial  $p$  such that  $p(M_z^*)$  is hypercyclic on  $\mathcal{H}$ .*

*Proof.* Choose  $n \geq 1$  such that  $G^n := \{z^n : z \in G\}$  satisfies  $G^n \cap \{z \in \mathbb{C} : \operatorname{Re}(z) < 1\} \neq \emptyset$ . To see how to do this, choose a polar rectangle  $R = \{re^{i\theta} : r_1 < r < r_2 \text{ and } \alpha < \theta < \beta\}$  such that  $R \subseteq G$ . Then simply choose a positive integer  $n$  such that  $n(\beta - \alpha) > 2\pi$ . Then  $R^n \subseteq G^n$  and  $R^n$  will contain the annulus  $\{re^{i\theta} : r_1^n < r < r_2^n\}$ , so certainly  $G^n \cap \{z \in \mathbb{C} : \operatorname{Re}(z) < 1\} \neq \emptyset$ . Now if  $0 < a \leq 1$ , then the convex polynomial  $p_a(z) = az + (1-a)$  maps the disk  $B(\frac{a-1}{a}, \frac{1}{a})$

onto the unit disk. Notice that the family of disks  $\{B(\frac{a-1}{a}, \frac{1}{a}) : 0 < a < 1\}$  is the family of all disks that are centered on the negative real axis and pass through the point  $z = 1$ . Thus it follows that  $\{z \in \mathbb{C} : \operatorname{Re}(z) < 1\} = \bigcup_{0 < a < 1} B(\frac{a-1}{a}, \frac{1}{a})$ . So we can choose an  $a \in (0, 1)$  such that  $G^n \cap \partial B(\frac{a-1}{a}, \frac{1}{a}) \neq \emptyset$ . It follows that the polynomial  $p(z) = p_a(z^n)$  is a convex polynomial and furthermore it satisfies  $p(G) \cap \partial\mathbb{D} \neq \emptyset$ .

Thus  $M_p^*$  is hypercyclic on  $\mathcal{H}$ . However,  $M_p^* = p^\#(M_z^*)$  where  $p^\#(z) = \overline{p(\bar{z})}$ . Also, since  $p$  is a convex polynomial, all of its coefficients are real, thus  $p^\# = p$ . Thus,  $p(M_z^*) = p^\#(M_z^*) = M_p^*$  is hypercyclic on  $\mathcal{H}$ .  $\square$

In the next result we give an example of an operator that is convex-cyclic but no convex polynomial of the operator is hypercyclic. In other words, the operator is purely convex-cyclic.

**Example 6.8.** Let  $\{\alpha_n\}_{n=1}^\infty$  and  $\{\beta_n\}_{n=1}^\infty$  be two strictly decreasing sequences of positive numbers that are interlaced and converging to zero. In other words,  $0 < \alpha_{n+1} < \beta_{n+1} < \alpha_n$  for all  $n \geq 1$  and  $\alpha_n \rightarrow 0$  (and hence  $\beta_n \rightarrow 0$ ). For each  $n \geq 1$ , let

$$G_n := \{re^{i\theta} : 2 < r < 2 + \frac{1}{n} \text{ and } \alpha_n < \theta < \beta_n\}.$$

Let  $G := \bigcup_{n=1}^\infty G_n$  and let  $L_a^2(G)$  be the Bergman space of all analytic functions on  $G$  that are square integrable with respect to area measure on  $G$ . Then the operator  $M_z^*$  is purely convex-cyclic on  $L_a^2(G)$ ; meaning that  $M_z^*$  is convex-cyclic on  $L_a^2(G)$ , but  $p(M_z^*)$  is not hypercyclic on  $L_a^2(G)$  for any convex polynomial  $p$ .

*Proof.* By Corollary 6.4 we know that  $M_z^*$  is convex-cyclic on  $L_a^2(G)$ . In order to show that no convex polynomial of  $M_z^*$  is hypercyclic, suppose, by way of contradiction, that there exists a convex polynomial  $p$  such that  $p(M_z^*)$  is hypercyclic. Since  $p$  is a convex polynomial it has real coefficients thus  $p^\#(z) = p(z)$  where  $p^\#(z) := \overline{p(\bar{z})}$ . Thus  $p(M_z^*) = M_{p^\#}^* = M_p^*$  and it follows that  $M_p^*$  is hypercyclic on  $L_a^2(G)$ . Thus it follows that every component  $G_n$  of  $G$  must satisfy that  $p(G_n) \cap \partial\mathbb{D} \neq \emptyset$ . However since  $p$  is a convex polynomial,  $p$  is (strictly) increasing on the interval  $[0, \infty)$ . Thus,  $p(2) > p(1) = 1$ . Choose an  $\varepsilon > 0$  such that  $\varepsilon < p(2) - 1$ . Since  $p$  is continuous at  $z = 2$ , and since we have an  $\varepsilon > 0$ , then there exists a  $\delta > 0$  such that if  $|z - 2| < \delta$ , then  $|p(z) - p(2)| < \varepsilon$ . Notice that for  $n$  sufficiently large we have that  $G_n \subseteq B(2, \delta)$ , thus,  $p(G_n) \subseteq B(p(2), \varepsilon) \subseteq \{z \in \mathbb{C} : \operatorname{Re}(z) > 1\}$ . Thus,  $p(G_n) \cap \partial\mathbb{D} = \emptyset$  for all large  $n$ , a contradiction. It follows that no convex polynomial of  $M_z^*$  is hypercyclic, hence  $M_z^*$  is purely convex-cyclic.  $\square$



## 7. OPEN QUESTIONS

It is well known that hypercyclic operators have a dense set of hypercyclic vectors. In fact, the set of hypercyclic vectors is a dense  $G_\delta$  set.

**Question 1.** *If  $T$  is convex-cyclic, then does  $T$  have a dense set of convex-cyclic vectors?*

Sanders [24] proved that if  $T : H \rightarrow H$  is a hyponormal operator on a Hilbert space  $H$ , then  $T$  is not weakly hypercyclic. A hyponormal operator is *pure* if its restriction to any of its reducing subspaces is not normal. That is, a hyponormal operator  $T$  is pure if  $T$  cannot be written in the form  $T = S \oplus N$  where  $N$  is a normal operator.

**Question 2.** Are there pure hyponormal operators or continuous normal operators that are convex-cyclic?

**Question 3.** *If  $T$  is convex-cyclic on a complex Hilbert space, then is  $(-1)T$  also convex-cyclic?*

The above question is true for diagonal normal operators/matrices and the other examples in this paper and also whenever  $T^2$  is convex-cyclic.

**Question 4.** If  $T$  is a convex-cyclic operator, then how big can the point spectrum of  $T^*$  be? Can it have non-empty interior?

Bourdon and Feldman [10] showed that if a vector  $x \in X$  has a somewhere dense orbit under a bounded linear operator  $T$ , then the orbit of  $x$  under  $T$  must be everywhere dense in  $X$ . A similar question was posed for convex-cyclicity by Rezaei. Recently, León-Saavedra and Romero de la Rosa provide an example where Bourdon and Feldman's result fails for convex-cyclic operators  $T$  such that  $\sigma_p(T^*) \neq \emptyset$ .

**Question 5.** [23, Question 5.5] Let  $X$  be a Banach space and  $T \in L(X)$  where  $\sigma_p(T^*) = \emptyset$ . If  $x \in X$  and  $co(Orb(T, x))$  is somewhere dense in  $X$ , then is  $co(Orb(T, x))$  dense in  $X$ ?

Since it is unknown if there exists a Banach space on which every hypercyclic operator is weakly mixing, we ask:

**Question 6.** *Given a separable Banach space  $X$ , is there a convex-cyclic operator  $S$  on  $X$  such that  $S^2$  is not convex-cyclic?*

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*E-mail address:* tbermude@ull.es

*E-mail address:* abonilla@ull.es

*E-mail address:* feldmanN@wlu.edu

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE LA LAGUNA, 38271 LA LAGUNA (TENERIFE), SPAIN

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE LA LAGUNA, 38271 LA LAGUNA (TENERIFE), SPAIN

DEPT. OF MATHEMATICS, WASHINGTON AND LEE UNIVERSITY, LEXINGTON VA 24450